

# PRIMITIVE PERMUTATION GROUPS AS PRODUCTS OF POINT STABILIZERS

MARTINO GARONZI, DAN LEVY, ATTILA MARÓTI, AND IULIAN I. SIMION

**ABSTRACT.** We prove that there exists a universal constant  $c$  such that any finite primitive permutation group of degree  $n$  with a non-trivial point stabilizer is a product of no more than  $c \log n$  point stabilizers.

## 1. INTRODUCTION

Given a finite group  $G$ <sup>1</sup> and a subgroup  $H$  of  $G$  whose normal closure is  $G$ , one can show, by a straightforward elementary argument, that  $G$  is the setwise product of at least  $\frac{\log|G|}{\log|H|}$  conjugates of  $H$ . A far reaching conjecture of Liebeck, Nikolov and Shalev states [8] that in the case that  $G$  is a non-abelian simple group,  $\frac{\log|G|}{\log|H|}$  is in fact the right order of magnitude for the minimal number of conjugates of  $H$  whose product is  $G$ , namely, there exists a universal constant  $c$  such that for any non-abelian simple group  $G$  and any non-trivial  $H \leq G$ , the group  $G$  is the product of no more than  $c \frac{\log|G|}{\log|H|}$  conjugates of  $H$ . Later on, in [9], this conjecture was extended to allow  $H$  to be any subset of  $G$  of size at least 2. Some weaker versions of these conjectures are proved in [8, Theorem 2], [9, Theorem 3], and [4, Theorem 1.3].

Here we look for a universal upper bound on the minimal length of a product covering of a finite primitive permutation group by conjugates of a point stabilizer. We will prove the following logarithmic<sup>2</sup> bound:

**Theorem 1.** *There exists a universal constant  $c$  such that if  $G$  is any primitive permutation group of degree  $n$  with a non-trivial point stabilizer  $H$  then  $G$  is a product of at most  $c \log n$  conjugates of  $H$ .*

Note that in most relevant cases,  $\frac{\log|G|}{\log|H|} < \log|G : H| = \log n$  (see Lemma 2.1). Thus we do not know whether the bound provided by Theorem 1 is the best possible. In fact, on the basis of currently published results we don't even know if this bound can be improved for any particular O'Nan-Scott family of primitive groups. We believe that these questions deserve further investigation.

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<sup>1</sup>All groups discussed are assumed to be finite.

<sup>2</sup>Throughout the paper,  $\log$  stands for logarithm in base 2.

## 2. PRELIMINARIES

We collect some preparatory results and notation.

**Lemma 2.1.** *Let  $G$  be a group and  $H \leq G$  such that  $|H| \geq 4$  and  $|G : H| \geq 4$ . Then  $\log |G| / \log |H| \leq \log |G : H|$ .*

*Proof.* Set  $x := \log |G|$  and  $y := \log |H|$ . Then the desired inequality reads  $x/y \leq x - y$ , which is equivalent to  $x \geq y + 1 + \frac{1}{y-1}$ . Since  $y \geq 2$  because  $|H| \geq 4$ , this is clearly satisfied if  $x \geq y + 2$ , which is equivalent to  $|G : H| \geq 4$ .  $\square$

**Lemma 2.2.** *Let  $G$  be an almost simple group with socle  $T$ . Let  $M$  be a maximal subgroup of  $G$  and let  $M_0 := T \cap M$ . Then  $|M_0| \geq 6$ .*

*Proof.* We can assume that  $T \not\leq M$ . Since  $G$  is almost simple, we have  $M_0 \neq 1$  ([2, Theorem 1.3.6]) whence  $|M_0| \geq 2$ . Moreover,  $M_0 \trianglelefteq M$ , so by maximality of  $M$ , the fact that  $T$  is simple, and  $1 < M_0 < T$ , we get that  $M = N_G(M_0)$  and  $M_0 = M \cap T = N_T(M_0)$ . Suppose, by contradiction, that  $2 \leq |M_0| \leq 5$ . Then  $M_0$  is contained in a Sylow  $p$ -subgroup  $P$  of  $T$  where  $p \in \{2, 3, 5\}$  according to the case. If  $M_0 < P$  then  $M_0 < N_P(M_0) \leq N_T(M_0) = M_0$  - a contradiction. Thus  $M_0$  is a Sylow  $p$ -subgroup of  $T$ . But  $2 \leq |M_0| \leq 5$  implies that  $M_0$  is abelian so  $M_0 \leq C_T(M_0) \leq N_T(M_0) = M_0$ . Thus  $M_0 = Z(N_T(M_0))$ , and by Burnside's  $p$ -complement theorem ([11, 10.21]),  $M_0$  has a normal  $p$ -complement in  $T$  - a contradiction since  $T$  is simple.  $\square$

The following lemma is an easy corollary to a major result of [5]. Let  $x^G$  denote the conjugacy class of  $x$  in  $G$ .

**Lemma 2.3.** *Let  $T$  be a non-abelian simple group. Then there exist  $\alpha, \beta \in T$  such that  $T = \alpha^T \beta^T S$ , where  $S$  is any subset of  $T$  of size at least 2. In particular, there exist  $\alpha, \beta \in T$  such that  $T = \alpha^T \beta^T \gamma^T$  where  $\gamma := \beta^{-1} \alpha^{-1}$ .*

*Proof.* By [5, Theorem 1.4] there exist  $\alpha, \beta \in T$  with  $\alpha^T \beta^T \cup \{1\} = T$ . If  $\alpha^T \beta^T = T$  then we are done. Otherwise,  $\alpha^T \beta^T = T - \{1\}$ , and since for any  $s \in T$  we have  $(T - \{1\})s = T - \{s\}$ , we get that for any  $s_1 \neq s_2 \in S$  we have  $\alpha^T \beta^T s_1 \cup \alpha^T \beta^T s_2 = T$  and  $T = \alpha^T \beta^T S$  follows. For proving  $T = \alpha^T \beta^T \gamma^T$  (for the same choice of  $\alpha, \beta \in T$ ) we can assume  $\alpha^T \beta^T = T - \{1\}$ . Hence  $\gamma \neq 1$ , implying  $|\gamma^T| \geq 2$ . Now  $T = \alpha^T \beta^T \gamma^T$  follows by taking  $S = \gamma^T$  in the first claim.  $\square$

**Notation 1.** *We denote by  $\gamma_{\text{cp}}^H(G)$  the minimal positive integer  $m$  such that there exist  $m$  conjugates of  $H \leq G$  whose product is  $G$  ( $\gamma_{\text{cp}}^H(G) = \infty$  if  $G$  is not a product of conjugates of  $H$ ).*

For the proof of Theorem 1 we use the classification of finite primitive permutation groups as given by the O'Nan-Scott theorem, for which we adopt the formulation and notation of [10]. Thus  $G$  is assumed to be a primitive permutation group on a set  $\Omega$  of size  $n = |G : H|$  where  $H = G_\alpha$  is the stabilizer of some  $\alpha \in \Omega$ . The socle of  $G$  is denoted  $B \cong T^k$  with  $k \geq 1$ , where  $T$  is a simple group. Since  $B$  acts transitively on  $\Omega$  (being a non-trivial normal subgroup of a primitive group), we have  $G = BG_\alpha = BH$ . Suppose that  $B$  is contained in the product of  $t$  conjugates of  $H$ . Then  $G$  is a product of  $t$  conjugates of  $H$  (see [3, Lemma 7(2)]). Moreover, since  $B_\alpha = B \cap H \leq H$ , we get that  $B$  is certainly contained in the product of  $\gamma_{\text{cp}}^{B_\alpha}(B)$  conjugates of  $H$ . These considerations show that  $\gamma_{\text{cp}}^{G_\alpha}(G) \leq \gamma_{\text{cp}}^{B_\alpha}(B)$  while

$n = |G : G_\alpha| = |B : B_\alpha|$  and so in the cases where  $B$  does not act regularly on  $\Omega$  we will prove our claim by exhibiting a suitable upper bound on  $\gamma_{\text{cp}}^{B_\alpha}(B)$  (if  $B$  acts regularly,  $B_\alpha = 1$  and  $\gamma_{\text{cp}}^{B_\alpha}(B) = \infty$ ). Also note that since for any integer  $m$  there are only finitely many isomorphism types of finite groups  $A$  such that  $|A| \leq m$ , for all primitive groups  $G$  satisfying  $|G| \leq m$  we get that  $\gamma_{\text{cp}}^{B_\alpha}(B)$  is bounded above by some constant depending on  $m$ , and so we may assume, that  $|B| = |T|^k > m$  for any fixed choice of  $m$ .

### 3. Type I. $G$ IS AN AFFINE PRIMITIVE PERMUTATION GROUP

**Proposition 3.1.** *Let  $G$  be an affine primitive permutation group with a non-trivial point stabilizer  $H$ . Then  $G$  is a product of at most  $1 + c_A \log |G : H|$  conjugates of  $H$ , where  $0 < c_A \leq 3/\log 5 < 1.3$  is a universal constant.*

In order to prove Proposition 3.1, we review some basic properties of affine primitive permutation groups. If  $G$  is an affine primitive permutation group, then it has exactly one minimal normal subgroup  $V$ , which is abelian so  $V \cong C_p^l$  for some prime  $p$  and some natural number  $l$ . Moreover  $G = VH$  and, viewing  $V$  as the vector space over  $\mathbb{F}_p$ , then  $H$  acts by conjugation irreducibly as a group of linear transformations on  $V$ . When convenient we will use additive notation for  $V$ .

**Lemma 3.2.** *Let  $G$  be an affine primitive permutation group with point stabilizer  $H$  and minimal normal subgroup  $V \cong C_p^l$ . Let  $h \in H$  and  $v \in V$ . Set  $w := v^{h^{-1}} - v$  and  $k := \lceil \log p \rceil$ . Then  $\langle w \rangle$  is contained in a product of  $k + 1$  conjugates of  $H$ .*

*Proof.* We can assume  $w \neq 0_V = 1_G$  for which the claim clearly holds. Then  $w$  is of order  $p$ , and any element of  $\langle w \rangle$  is of the additive form  $sw$  where the integer  $s$  satisfies  $0 \leq s \leq p - 1$ . Since  $k := \lceil \log p \rceil$ , the base 2 representation of  $s$  takes the form  $s = \sum_{j=0}^{k-1} b_j 2^j$  ( $b_j \in \{0, 1\}$  for all  $0 \leq j \leq k - 1$ ). Now note that  $w = v^{h^{-1}} - v = v^{-1}hv h^{-1} \in H^v H$ . Similarly, for any  $c \in \mathbb{F}_p$  we have  $cw = (cv)^{h^{-1}} - cv \in H^{cv} H$ . Thus, identifying the powers  $2^j$  with elements of  $\mathbb{F}_p$ , we see that  $sw \in (H^v H) (H^{2^v} H) (H^{2^{2^v} H}) \dots (H^{2^{k-1}v} H)$ , for any  $0 \leq s \leq p - 1$ , where we pick  $0_V$  from the  $j$ -th factor  $(H^{2^j v} H)$  in the product if  $b_j = 0$  and  $2^j w$  if  $b_j = 1$ . However, also note that since  $V$  is abelian,  $(H^{2^j v} H) \cap V$  is invariant under conjugation by any element of  $V$ . Hence, for any choice of  $u_0, \dots, u_{k-2} \in V$  we have

$$sw \in \Pi_H := (H^v H)^{u_0} (H^{2^v} H)^{u_1} (H^{2^{2^v} H})^{u_2} \dots (H^{2^{k-2}v} H)^{u_{k-2}} (H^{2^{k-1}v} H).$$

Finally, for the choice  $u_{k-2} = 2^{k-1}v$ ,  $u_{k-3} = u_{k-2} + 2^{k-2}v$  and in general  $u_{k-j} = u_{k-j+1} + 2^{k-j+1}v$  for all  $2 \leq j \leq k$  where  $u_{k-1} := 0_V$ , we get that  $\Pi_H$  is equal to a product of  $k + 1$  conjugates of  $H$ .  $\square$

**Lemma 3.3.** *For each prime number  $p$  define  $f(p) := \lceil \log p \rceil / \log p$ . Then  $f(p)$  has a global maximum at  $p = 5$ . Consequently*

$$(3.1) \quad \lceil \log p \rceil \leq (3/\log 5) \log p, \text{ for every prime } p.$$

*Proof.* First check that  $1 + 1/\log 11 < 1.29 < 3/\log 5$ . Then, using this, we get:

$$f(p) \leq (\log p + 1)/\log p = 1 + 1/\log p < 3/\log 5 = f(5), \forall p \geq 11,$$

and for  $p = 2, 3, 7$  we verify explicitly that  $f(p) < f(5)$ . Hence  $f(p)$  has a global maximum  $f(5) = 3/\log 5$  at  $p = 5$ . Finally,  $\lceil \log p \rceil = f(p) \log p \leq f(5) \log p$ .  $\square$

**Proof of Proposition 3.1.** Using the notation introduced after the statement of the proposition,  $\log |G : H| = \log |V| = \log p^l = l \log p$ . Using Inequality 3.1, we obtain:

$$1 + \lceil \log p \rceil \leq 1 + (3/\log 5)l \log p = 1 + (3/\log 5) \log |G : H|.$$

Thus, it is enough to show that  $G$  is a product of at most  $1 + \lceil \log p \rceil$  conjugates of  $H$ .

Fix a non-zero vector  $v \in V$ . If  $v$  is central in  $G$  then  $V = \langle v \rangle$  by minimality of  $V$ . It follows that  $H$  is a non-trivial normal subgroup of  $HV = G$  since  $V$  is central - a contradiction to  $H$  being core-free. Therefore  $v$  is not central, and there is some  $h \in H$  with  $v^{h^{-1}} \neq v$ . Set  $w := v^{h^{-1}} - v$ .

We claim that there are  $l$  elements  $h_1, \dots, h_l \in H$  such that  $B := \{w^{h_1}, \dots, w^{h_l}\}$  is a vector space basis of  $V = C_p^l$ . Note that since  $w \neq 0_V$ , this claim is immediate for  $l = 1$ , and hence we assume  $l \geq 2$ . Suppose by contradiction that  $1 \leq m < l$  is the maximal integer such that there exist  $h_1, h_2, \dots, h_m \in H$  for which  $B = \{w^{h_1}, \dots, w^{h_m}\}$  is linearly independent. It follows that for any  $h \in H$ ,  $w^h \in \text{Span}(B)$ . Thus  $\text{Span}(B) = \text{Span}(\{w^h | h \in H\})$ . This shows that  $\text{Span}(B)$  is a proper non-trivial  $H$ -invariant subspace of  $V$ , contradicting the fact that  $H$  acts irreducibly on  $V$ . Thus there exists a basis of  $V$  of the form  $B := \{w^{h_1}, \dots, w^{h_l}\}$ .

For each  $v \in V$  there exist  $s_1, \dots, s_l \in \mathbb{F}_p$  for which  $v = \sum_{i=1}^l s_i w^{h_i}$ . Applying Lemma 3.2 to each  $w^{h_i}$  separately, we get that each  $v \in V$  belongs to  $\Pi_1 \cdots \Pi_l$ , where each  $\Pi_i$  is a product of  $\lceil \log p \rceil + 1$  conjugates of  $H$ . But, as in the proof of Lemma 3.2, this shows that  $V \subseteq \Pi_1^{u_1} \cdots \Pi_{l-1}^{u_{l-1}} \Pi_l$  for any choice of  $u_1, \dots, u_{l-1} \in V$ , and one can choose these elements so that the product  $\Pi_1^{u_1} \cdots \Pi_{l-1}^{u_{l-1}} \Pi_l$  is a product of at most  $l \lceil \log p \rceil + 1$  conjugates of  $H$ .  $\square$

From Proposition 3.1 it follows that if  $G$  is an affine primitive permutation group then  $\gamma_{\text{cp}}^H(G) \leq c_I \log n$  where the constant  $c_I$  satisfies  $0 < c_I < 2.3$ .

#### 4. Type II. $G$ IS AN ALMOST SIMPLE PRIMITIVE PERMUTATION GROUP

In this case we have  $k = 1$  and  $B = T$ . Note that  $T$  is a non-abelian simple group acting transitively on  $\Omega$ . Furthermore,  $T$  does not act regularly on  $\Omega$  by [10].

First suppose that  $|G| < n^9$ . By [9, Theorem 3], since  $T_\alpha$  is a subset of  $T$  of size at least 2 (because  $T$  does not act regularly), there exists a constant  $c_1$  such that  $T$  is a product of less than  $c_1 \log |T|$  conjugates of  $T_\alpha$ . Now  $|T| \leq |G| < n^9$  implies that  $\gamma_{\text{cp}}^{T_\alpha}(T) < 9 \cdot c_1 \log n$ .

Assume that  $|G| \geq n^9$ . By [7] one of the following holds:

- (1)  $T = A_m$ , where  $m \geq 5$  and either
  - (a)  $\Omega$  is the set of all subsets of size  $k$  of  $\{1, \dots, m\}$ ,  $n = \binom{m}{k}$  or
  - (b)  $\Omega$  is the set of all partitions of  $\{1, \dots, m\}$  into  $a$  subsets of size  $b$  where  $ab = m$ ,  $a > 1$ ,  $b > 1$ ;  $n = m! / ((b!)^a a!)$ .

- (2)  $T$  is a classical simple group acting on an orbit of subspaces of the natural module, or (in the case  $T = PSL(d, q)$ ) on pairs of subspaces of complementary dimensions.

Since  $n = |G : G_\alpha| = |T : T_\alpha|$ , and since  $G$  is almost simple, we have by [1, Lemma 2.7 (i)] that  $|G : T| \leq |\text{Out}(T)| < n$ . This gives  $|T| > n^8 = \left(\frac{|T|}{|T_\alpha|}\right)^8$ , implying  $\frac{\log|T|}{\log|T_\alpha|} < \frac{8}{7} < 2$ . If  $T_\alpha$  is maximal in  $T$ , we can conclude from [8, Theorem 2] that there exist a universal constant  $c_2$  and a universal function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , such that for all  $T$  satisfying  $|T| > f(2)$  it holds that  $\gamma_{\text{cp}}^{T_\alpha}(T) \leq c_2 \frac{\log|T|}{\log|T_\alpha|}$ . Now we claim that this conclusion is in fact valid even if  $T_\alpha$  is not maximal in  $T$ . More precisely, we claim that [8, Theorem 2] is valid for all subgroups belonging to the families listed in [8, Lemma 3.1] in the case  $T = A_m$ , and in [8, Lemma 4.3] in the case that  $T$  is a classical group. Note that these families include the  $(T, T_\alpha)$  of [7] listed above. Our claim is based on a close examination of the use of the maximality assumption in the proof of [8, Theorem 2]. We find that the maximality assumption is used only in two places. First, in appealing to [8, Theorem 1] in order to discard cases of simple groups of Lie type of small Lie rank. Here we replace [8, Theorem 1] by [4, Theorem 1.3], which applies to any subset of  $T$  of size at least 2. The second use of the maximality assumption is to identify the possible isomorphism types for maximal subgroups of the remaining simple groups, according to the O’Nan-Scott classification in the alternating case and the Aschbacher classification in the classical case. These are precisely the families listed in [8, Lemma 3.1] and in [8, Lemma 4.3]. The rest of the proof of Theorem 2 of [8] carries through even when the subgroups in question are not actually maximal.

Finally, by Lemma 2.1 and Lemma 2.2 we get that  $\gamma_{\text{cp}}^H(G) \leq c_{II} \log n$  for some universal constant  $c_{II} > 0$ , for all primitive almost simple  $G$ .

### 5. Type III(a). $G$ IS A PRIMITIVE PERMUTATION GROUP OF DIAGONAL TYPE.

Here  $B_\alpha$  is the diagonal subgroup of  $B$  ( $\Delta$  in the notation of Proposition 5.1 below) and  $n = |G : G_\alpha| = |T|^{k-1}$ , where  $k \geq 2$ .

**Proposition 5.1.** *Let  $T$  be a non-abelian simple group,  $k$  a positive integer,  $B := T^k$ . Set  $\Delta := \{(t, t, \dots, t) : t \in T\} \leq B$ . Then  $k \leq \gamma_{\text{cp}}^\Delta(B) \leq 3k - 2$ .*

*Proof.* Suppose  $B$  is a product of  $m$  conjugates of  $\Delta$ . Then  $\Delta \cong T$  implies  $|T|^k = |B| \leq |\Delta|^m = |T|^m$ . This proves  $k \leq \gamma_{\text{cp}}^\Delta(B)$ . For proving  $\gamma_{\text{cp}}^\Delta(B) \leq 3k - 2$ , choose  $\alpha, \beta, \gamma \in T$  as in Lemma 2.3. Set  $a := \alpha^{-1}$  and  $b := \gamma$ . Then  $T = \alpha^T \beta^T \gamma^T = (a^{-1})^T (ab^{-1})^T b^T$ .

Let  $i \in \{1, \dots, k\}$ . Let  $\tau_i : T \rightarrow T^k$  be the map that sends  $t \in T$  to the element of  $T^k$  that has  $t$  in the  $i$ -th component and 1 elsewhere. We denote  $T_i := \tau_i(T)$ . Consider  $D_i := \Delta \Delta^{\tau_i(a)} \Delta^{\tau_i(b)} \Delta$ . We prove that  $D_i \supseteq T_i$ . An element of  $D_i$  has the form

$$(xyzw, xyzw, \dots, xyzw, \underbrace{xy^a z^b w}_{i\text{-th entry}}, xyzw, \dots, xyzw)$$

where  $x, y, z, w \in T$  are arbitrary. In order to prove that  $D_i \supseteq T_i$ , choose arbitrary  $x, y, z \in T$  and  $w = (xyz)^{-1}$ . Then for the  $i$ -th component we have

$$\begin{aligned} xy^a z^b w &= xy^a z^b (xyz)^{-1} = xa^{-1} yab^{-1} zbz^{-1} y^{-1} x^{-1} \\ &= (xa^{-1} x^{-1})((xy)(ab^{-1})(y^{-1} x^{-1}))((xyz)b(z^{-1} y^{-1} x^{-1})) \\ &\in (a^{-1})^T (ab^{-1})^T b^T = T. \end{aligned}$$

Since  $\{(x, xy, xyz) \mid x, y, z \in T\} = T^3$ , we can deduce  $D_i \supseteq T_i$ .

It follows that  $B = T_1 \cdots T_k = \Delta T_2 \cdots T_k \subseteq \Delta D_2 \cdots D_k = D_2 \cdots D_k$ . Therefore

$$\begin{aligned} B &= D_2 \cdots D_k = (\Delta \Delta^{\tau_2(a)} \Delta^{\tau_2(b)} \Delta) \cdots (\Delta \Delta^{\tau_k(a)} \Delta^{\tau_k(b)} \Delta) \\ &= (\Delta \Delta^{\tau_2(a)} \Delta^{\tau_2(b)}) \cdots (\Delta \Delta^{\tau_k(a)} \Delta^{\tau_k(b)}) \Delta, \end{aligned}$$

and  $B$  is a product of  $3(k-1) + 1 = 3k - 2$  conjugates of  $\Delta$ .  $\square$

By Proposition 5.1 we have  $\gamma_{\text{cp}}^\Delta(B) \leq 3k - 2$ . On the other hand

$$\log |G : G_\alpha| = (k-1) \log |T| \geq (k-1) \log 60 > 5(k-1).$$

Comparing the numbers we see that  $B$  is the product of less than  $\log |G : G_\alpha|$  conjugates of  $B_\alpha$ , and so we have  $\gamma_{\text{cp}}^H(G) \leq c_{III(a)} \log n$  with  $0 < c_{III(a)} \leq 1$ .

## 6. Type III(b). $G$ IS A PRIMITIVE PERMUTATION GROUP OF PRODUCT ACTION TYPE.

Let  $R$  be a primitive permutation group of type II or III(a) on a set  $\Gamma$ . For  $\ell > 1$ , let  $W = R \wr S_\ell$ , and take  $W$  to act on  $\Omega = \Gamma^\ell$  in its natural product action. Then for  $\gamma \in \Gamma$  and  $\alpha = (\gamma, \dots, \gamma) \in \Omega$  we have  $W_\alpha = R_\gamma \wr S_\ell$ , and  $n = |\Gamma|^\ell$ . If  $K$  is the socle of  $R$  then the socle  $B$  of  $W$  is  $K^\ell$ , and  $B_\alpha = (K_\gamma)^\ell \neq 1$ . If  $G$  is primitive of type III(b), then  $G$  satisfies  $B \leq G \leq W$  and acts transitively on the  $\ell$  factors of  $B = K^\ell$ . In particular,  $\text{soc}(G) = \text{soc}(W) = K^\ell$ . By the discussion of cases II and III(a) we know that  $K$  is the product of at most  $\max\{c_{II}, c_{III(a)}\} \cdot \log |K : K_\gamma|$  conjugates of  $K_\gamma$ . Since  $B = K^\ell$  and  $B_\alpha = (K_\gamma)^\ell$ , we get that  $B$  is the product of at most  $\max\{c_{II}, c_{III(a)}\} \cdot \log |K : K_\gamma|$  conjugates of  $B_\alpha$ . Now  $|G : G_\alpha| = |\Gamma|^\ell$ , and, since  $K$  acts transitively on  $\Gamma$ ,  $|\Gamma| = |K : K_\gamma|$ . Hence

$$\log n = \log |G : G_\alpha| = \log |\Gamma|^\ell = \ell \log |K : K_\gamma|,$$

and we have proved that  $\gamma_{\text{cp}}^H(G) \leq c_{III(b)} \log n$  with  $0 < c_{III(b)} \leq \max\{c_{II}, c_{III(a)}\}$ .

## 7. Type III(c). $G$ IS A PRIMITIVE PERMUTATION GROUP OF TWISTED WREATH PRODUCT TYPE.

Let  $P$  be a transitive permutation group of degree  $k$ , acting on  $\{1, \dots, k\}$ , and let  $Q \leq P$  be the stabilizer of 1. Let  $\varphi : Q \rightarrow \text{Aut}(T)$  be a homomorphism such that  $\varphi(Q)$  contains all the inner automorphisms of  $T$ . Let

$$B_0 = \{f : P \rightarrow T : f(pq) = f(p)^{\varphi(q)} \forall p \in P, q \in Q\}.$$

Then  $B_0$  is a group with pointwise multiplication. let  $L = \{l_1, \dots, l_k\} \subseteq P$  be an arbitrary fixed left transversal of  $Q$  in  $P$ . By definition of  $B_0$ , a function  $f \in B_0$  is determined by its values on  $L$ . On the other hand, the values of  $f$  on  $L$  can be

arbitrary, and therefore we get  $B_0 \cong T^k$ . More specifically, for  $\ell \in L$  and  $t \in T$  define  $f_{t,\ell} : P \rightarrow T$  by:

$$f_{t,\ell}(x) := \begin{cases} t^{\varphi(\ell^{-1}x)} & \text{if } x \in \ell Q, \\ 1 & \text{if } x \notin \ell Q \end{cases} \quad \forall x \in P.$$

We claim that  $f_{t,\ell} \in B_0$ . Indeed, let  $p \in P$ ,  $q \in Q$ , and consider  $f_{t,\ell}(pq)$ . If  $pq \notin \ell Q$  then  $p \notin \ell Q$  and  $f(pq) = f(p) = 1$  so  $f(pq) = f(p)^{\varphi(q)}$  holds. If  $pq \in \ell Q$  then  $p \in \ell Q$  and there exists  $q_0 \in Q$  such that  $p = \ell q_0$ . Hence

$$\begin{aligned} f_{t,\ell}(pq) &= f_{t,\ell}(\ell q_0 q) = t^{\varphi(\ell^{-1}\ell q_0 q)} = t^{\varphi(q_0 q)} = t^{\varphi(q_0)\varphi(q)} \\ &= \left(t^{\varphi(q_0)}\right)^{\varphi(q)} = f_{t,\ell}(\ell q_0)^{\varphi(q)} = f(p)^{\varphi(q)}. \end{aligned}$$

Furthermore, if  $\ell = l_i$  then  $f_{t,\ell}$  corresponds to the element of  $T^k$  that has  $t$  in the  $i$ -th component and 1 elsewhere. To see this we just have to check that  $f_{t,l_i}(l_j)$  satisfies  $f_{t,l_i}(l_j) = t$  if  $i = j$  and  $f_{t,l_i}(l_j) = 1$  if  $i \neq j$  and this is immediate from the definition. Thus we can construct an explicit isomorphism  $B_0 \rightarrow T^k$  which maps  $\{f_{t,l_i} | t \in T\}$  onto  $T_i$ , where  $T_i$  is the  $i$ -th direct factor of  $T^k$ ,  $1 \leq i \leq k$ . From now on we identify  $\{f_{t,l_i} | t \in T\}$  with  $T_i$ . Furthermore,  $P$  acts on  $B_0$  in the following way: if  $f \in B_0$  and  $p \in P$  define  $f^p(x) := f(px)$  for all  $x \in P$ . The semidirect product  $G := B_0 \rtimes P$  with respect to this action is called the twisted wreath product (of  $P$  and  $T$ ). Then  $G$  acts transitively by right multiplication on the set  $\Omega$  of size  $n = |B_0|$  of all right cosets of  $P$ . This action is not always primitive. If it is,  $G$  belongs to class III(c). In this case  $B = B_0 \cong T^k = T_1 \times \cdots \times T_k$  is a normal subgroup (the unique minimal one) in  $G$  and acts regularly on  $\Omega$ , and we take  $G_\alpha = P$ . We have  $n = |T|^k$ .

Set, for each  $1 \leq i \leq k$ ,  $Q_i := l_i Q l_i^{-1}$ . We prove that  $Q_i$  leaves  $T_i$  invariant with respect to the action of  $P$  on  $B_0$ . For this we have to show that if  $p \in Q_i$ , namely,  $p = l_i q l_i^{-1}$  for some  $q \in Q$ , then, for all  $x \in P$ ,  $x \in l_i Q$  if and only if  $px \in l_i Q$ . But  $px \in l_i Q$  means  $l_i q l_i^{-1} x \in l_i Q$ , and this is true if and only if  $q l_i^{-1} x \in Q$ , which, since  $q \in Q$ , is equivalent to  $l_i^{-1} x \in Q$ , which is equivalent to  $x \in l_i Q$ .

Thus the action of  $P$  on  $B_0$  induces an action of  $Q_i$  on  $T_i$  for each  $1 \leq i \leq k$ . Note that for  $p = l_i q l_i^{-1} \in Q_i$  and  $x = l_i q_0 \in l_i Q$ , we get

$$\begin{aligned} f_{t,l_i}^p(x) &= f_{t,l_i}(px) = t^{\varphi(l_i^{-1}px)} = t^{\varphi(l_i^{-1}l_i q l_i^{-1}l_i q_0)} \\ &= t^{\varphi(qq_0)} = \left(t^{\varphi(q)}\right)^{\varphi(q_0)} = f_{t^{\varphi(q)},l_i}(x). \end{aligned}$$

Since  $f_{t,l_i}^p(x) = f_{t^{\varphi(q)},l_i}(x)$  clearly holds also for any  $x \notin l_i Q$  we get  $f_{t,l_i}^p = f_{t^{\varphi(q)},l_i}$  and so  $Q_i$  acts on  $T_i$  as  $\varphi(Q)$ .

By Lemma 2.3 there exist  $t_{i,1}, t_{i,2}, t_{i,3} \in T_i$  for each  $1 \leq i \leq k$  such that  $T_i = t_{i,1}^{T_i} t_{i,2}^{T_i} t_{i,3}^{T_i}$ . For each  $j \in \{1, 2, 3\}$  set  $O_{i,j} := \{p^{-1} t_{i,j} p | p \in Q_i\}$ . In  $G := B_0 \rtimes P$ , the action of  $P$  on  $B_0$  is the restriction of the conjugation action of  $G$  on itself, and hence  $O_{i,j}$  is the orbit of  $t_{i,j}$  under the action of  $Q_i$  on  $T_i$ . Since, by assumption,  $\text{Inn}(T) \leq \varphi(Q)$ ,  $O_{i,j}$  is a normal subset of  $T_i$ , and in particular contains  $t_{i,j}^{T_i}$ , the conjugacy class of  $t_{i,j}$  in  $T_i$ . Set  $X_{i,j} := t_{i,j}^{-1} O_{i,j}$ ,  $1 \leq j \leq 3$ . Since the  $O_{i,j}$ 's are normal sets,

$$T_i = t_{i,1}^{-1} t_{i,2}^{-1} t_{i,3}^{-1} t_{i,1}^{T_i} t_{i,2}^{T_i} t_{i,3}^{T_i} \subseteq t_{i,1}^{-1} t_{i,2}^{-1} t_{i,3}^{-1} O_{i,1} O_{i,2} O_{i,3} = X_{i,1} X_{i,2} X_{i,3}.$$

Moreover  $X_{i,j} = \{t_{i,j}^{-1}p^{-1}t_{i,j}p : p \in Q_i\} \subseteq P^{t_{i,j}}P$  for  $j = 1, 2$ , and  $X_{i,3} = t_{i,3}^{-1}O_{i,3} = O_{i,3}t_{i,3}^{-1} = \{p^{-1}t_{i,3}pt_{i,3}^{-1} : p \in Q_i\} \subseteq PP^{t_{i,3}^{-1}}$ . It follows that

$$T_i = X_{i,1}X_{i,2}X_{i,3} \subseteq P^{t_{i,1}}PP^{t_{i,2}}PP^{t_{i,3}^{-1}}.$$

Thus  $B = T_1 \cdots T_k$  is contained in a product of at most  $5k$  conjugates of  $P$ , and since  $n = |T|^k$  we have  $k = \log n / \log |T|$ . Hence

$$\gamma_{\text{cp}}^P(G) \leq 5k = 5 \log n / \log |T| \leq \frac{5}{\log 60} \log n < \log n.$$

We have proved that if  $G$  is a primitive group of twisted wreath product type then  $\gamma_{\text{cp}}^H(G) \leq c_{III(c)} \log n$  where  $0 < c_{III(c)} < 1$ .

This completes the proof of Theorem 1.

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(Martino Garonzi) DEPARTAMENTO DE MATEMATICA, UNIVERSIDADE DE BRASÍLIA, CAMPUS UNIVERSITÁRIO DARCY RIBEIRO, BRASÍLIA - DF 70910-900, BRASIL  
*E-mail address:* mgaronzi@gmail.com

(Dan Levy) THE SCHOOL OF COMPUTER SCIENCES, THE ACADEMIC COLLEGE OF TEL-AVIV-YAFFO, 2 RABENU YERUHAM ST., TEL-AVIV 61083, ISRAEL  
*E-mail address:* danlevy@mta.ac.il

(Attila Maróti) MTA ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, REÁLTANODA UTCA 13-15, H-1053, BUDAPEST, HUNGARY  
*E-mail address:* maroti.attila@renyi.mta.hu

(Iulian I. Simion) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY  
*E-mail address:* iulian.simion@math.unipd.it